Disentangling Entanglement Entropy

Measuring Entanglement in 2D CFTs

David Grabovsky

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Motivation and Punchline

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A: In general, this is hard. But in 2D CFT, one can answer the last question precisely. The answer is the following formula:

Entanglement entropy of an interval in 2D CFT:

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Today, I will explain what this means and where it comes from.

Dramatis personæ: density matrices; Rényi entropies; path integrals; the replica trick; Riemann surfaces; twist fields; uniformization; conformal maps; the conformal Ward identity.

Outline

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1. Entanglement Entropy in QFT
2. The Replica Trick
   - Basic Setup and Path Integrals
   - Riemann Surfaces and Twist Fields
3. Conformal Shenanigans
   - Uniformizing the Stress Tensor
   - The Twist Field Scaling Dimension
4. A Few Generalizations

The von Neumann Entropy

Let $\mathcal{H}$ be the Hilbert space of a quantum system with Hamiltonian H. The partition function is $Z(\beta) = \mathrm{Tr}\left[e^{-\beta H}\right]$, for $\beta \in \mathbb{R}_+$.

Definition (Density operator)

A **density operator** ρ is a positive, Hermitian operator on $\mathcal{H}$ with $\mathrm{Tr}[\rho] = 1$. If ρ has rank 1, then $\rho = |\Psi\rangle\langle\Psi|$ is called a **pure state**. If ρ has higher rank, then ρ is called a **mixed state**.

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The **thermal state** at temperature $T = \frac{1}{\beta}$ is $\rho_{\mathrm{th}} = \frac{1}{Z} e^{-\beta H}$. If H has a unique ground state $|\Psi_0\rangle$, then $\lim_{\beta\to\infty} \rho_{\mathrm{th}} = |\Psi_0\rangle\langle\Psi_0|$.

Definition (von Neumann Entropy)

The **von Neumann entropy** of ρ is given by

$$S_{\mathrm{vN}}[\rho] \equiv -\mathrm{Tr}[\rho \log(\rho)] = -\sum_{\lambda\in\sigma(\rho)} \lambda \log(\lambda). \tag{1.1}$$

Entanglement Entropy

Divide the system into two parts, A and B: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. To describe entanglement between A and B, we "trace out" B:

Definition (Reduced density operator)

Given a density operator ρ on $\mathcal{H}$, the **reduced density operator** on $\mathcal{H}_A$ is $\rho_A \equiv \mathrm{Tr}_B[\rho]$, where the trace is taken over $\mathcal{H}_B$.

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If $\rho = |\Psi\rangle\langle\Psi|$ is pure, then ρ_A is pure only if $|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$ is a product state. The degree to which ρ_A is mixed captures the entanglement between A and B. This can be measured by S_{vN}:

Definition (Entanglement entropy)

The **entanglement entropy** of ρ on A is the vN entropy of ρ_A:

$$S_A[\rho] \equiv S_{\mathrm{vN}}[\rho_A] = -\mathrm{Tr}[\rho_A \log(\rho_A)] = -\mathrm{Tr}_A[\rho_A \log(\rho_A)]. \quad (1.2)$$

The Rényi Entropies

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Let $n \in \mathbb{Z}_+$. The n^{th} **Rényi entropy** is defined by

$$S_A^{(n)}[\rho] \equiv \frac{1}{1-n} \log \left[\mathrm{Tr}_A(\rho_A^n) \right]. \tag{1.3}$$

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An uninteresting calculation shows that

$$\lim_{n \to 1^+} S_A^{(n)}[\rho] = -\frac{\partial}{\partial n} \left[\mathrm{Tr}_A(\rho_A^n) \right]_{n=1} = S_A. \tag{1.4}$$

We aim to compute $\mathrm{Tr}_A(\rho_A^n)$ for $n \in \mathbb{Z}_+$ and then take $n \longrightarrow 1$.

N.B. Since the eigenvalues λ_i of ρ_A lie on $[0,1]$ and sum to one, the sum $\mathrm{Tr}_A(\rho_A^n) = \sum_i \lambda_i^n$ is absolutely convergent for *any* $n \in \mathbb{C}$ with $\mathrm{Re}\{n\} \geq 1$. Thus so is $S_A^{(n)}$; hence $\lim_{n \to 1^+} S_A^{(n)}$ is well-defined.

Preview of the Calculation

Step –1: set up a 2D CFT on $\mathbb{C}$. Here $A = [u, v]$ is an interval, and the entangling surface $\partial A = \{u, v\}$ consists of two points.

Step 0: Express Z, ρ, and ρ_A as path integrals with different BC.

Step 1: Write $\mathrm{Tr}_A(\rho_A^n) = \frac{Z(\mathcal{R}_n)}{Z^n}$ as a path integral on an n-sheeted Riemann surface $\mathcal{R}_n$, which implements the BC.

Step 2: Write $\mathrm{Tr}_A(\rho_A^n) = \langle \Phi_n \overline{\Phi}_n \rangle$ as a PI for the 2-point function of defect-like "twist fields" Φ_n, defined in the replica theory on $\mathbb{C}$.

Step 3: Steps 1 and 2 yield a dictionary between $\mathcal{R}_n$ and $\mathbb{C}$.

Preview of the Calculation

Step 4: Consider the stress tensor T. Use the transformation properties of T under a conformal *uniformizing map* to compute $\langle T\rangle_{\mathcal{R}_n}$. By the dictionary above, this is equal to $\frac{\langle \Phi_n \overline{\Phi}_n T\rangle_{\mathbb{C}}}{\langle \Phi_n \overline{\Phi}_n\rangle_{\mathbb{C}}}$.

Step 5: Use the conformal Ward identity to relate $\langle \Phi_n \overline{\Phi}_n T\rangle_{\mathbb{C}}$ to $\langle \Phi_n \overline{\Phi}_n\rangle_{\mathbb{C}}$, and thereby obtain the scaling dimension Δ_n of Φ_n.

Step 6: With Δ_n in hand, calculate $\mathrm{Tr}_A(\rho_A^n)$, find $S_A^{(n)}[\rho]$, and take $n \longrightarrow 1$ to find $S_A = \frac{c}{3}\log\left(\frac{\ell}{a}\right)$, as promised.

Step 7: Discuss generalizations and provide a bulk interpretation.

Outline

Setup: 2D QFT on a Lattice

Consider an infinite 1D lattice $\Lambda \subset \mathbb{C} = \{x + i\tau\}$ with spacing a on the real axis $\tau = 0$. The Hilbert space is $\mathcal{H} = \bigotimes_{x\in\Lambda} \mathcal{H}_x$. Let $\{\hat{\phi}_x^i\}$ be a complete set of commuting observables with spectrum $\{\phi_x^i\}$.

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The eigenvectors of the $\hat{\phi}^i_x$ form a basis for $\mathcal{H}$, and take the form

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The Hamiltonian is H, the partition function is $Z = \mathrm{Tr}\left[e^{-\beta H}\right]$, and the equilibrium state of the system is $\rho = \rho_{\text{th}} = \frac{1}{Z} e^{-\beta H}$.

We suppress the species i and will soon take $a \longrightarrow 0$, $\beta \longrightarrow \infty$.

The Partition Function

The partition function may be expressed as a path integral:

$$Z(\beta) = \sum_{\{\phi_y\}} \left\langle \prod_{y\in\Lambda} \phi_y \middle| e^{-\beta H} \middle| \prod_{y\in\Lambda} \phi_y \right\rangle =$$

$$= \oint_{\phi_y(0)=\phi_x}^{\phi_y(\beta)=\phi_x} \mathcal{D}\phi_y(\tau)\, e^{-S_{\mathrm{E}}}, \quad S_{\mathrm{E}} = \int_0^\beta \mathrm{d}\tau\, L_{\mathrm{E}}[\{\phi_x\}]. \quad (2.2)$$

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We "geometrize" the periodic BC by viewing the $\{\phi_x\}$ as living on the cylinder $\mathcal{C} = S^1_\tau(\beta) \times \Lambda$, which unfurls into $\mathbb{C}$ as $\beta \longrightarrow \infty$. So:

$$Z(\beta) = \int_{\mathcal{C}} \mathcal{D}\phi_y(\tau)\, e^{-S_{\mathrm{E}}}, \qquad S_{\mathrm{E}} = \int_{S^1_\tau(\beta)} \mathrm{d}\tau\, L_{\mathrm{E}}[\{\phi_x\}]. \quad (2.3)$$

The Density Operator

The matrix elements of ρ also have path integral expressions:

$$\rho(\phi_x, \phi'_{x'}) = \frac{1}{Z}\left\langle \prod_{x\in\Lambda} \phi_x \middle| e^{-\beta H} \middle| \prod_{x'\in\Lambda} \phi'_{x'} \right\rangle = \frac{1}{Z}\int_{\phi_y(0)=\phi'_{x'}}^{\phi_y(\beta)=\phi_x} \mathcal{D}\phi_y(\tau)\, e^{-S_{\mathrm{E}}}. \tag{2.4}$$

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Whereas Z is "sewn," ρ is "distended". Can we do a bit of both?

Let $A = [u, v]$ be the interval from $(u, 0)$ to $(v, 0)$, and $B = \overline{A}$. The path integral for $\rho_A = \mathrm{Tr}_B(\rho)$ identifies ϕ_x *only* along B.

The Reduced Density Operator

Thus the reduced density matrix is a path integral too:

$$\rho_A(\phi_x, \phi'_{x'}) = \frac{1}{Z} \int_{\phi_y(0)}^{\phi_y(\beta)} \mathcal{D}\phi_y(\tau)\, e^{-S_{\mathrm{E}}},$$

$$\mathrm{BC}: \begin{cases} \phi_y(0) = \phi_x, & y \notin A = [u, v], \\ \phi_y(0) = \phi'_{x'}, & y \in A = [u, v], \\ \phi_y(\beta) = \phi_x, & y \in \mathbb{R}_x. \end{cases} \tag{2.5}$$

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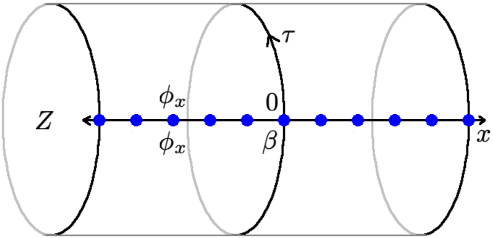

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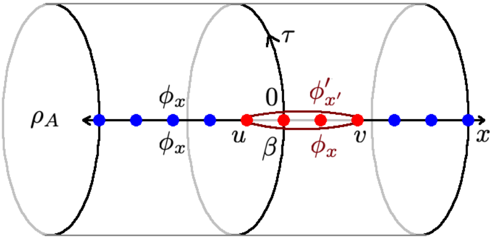

The Passage to Field Theory

In the continuum limit $a \longrightarrow 0$, we change notation slightly:

- $\Lambda \longrightarrow \mathbb{R}_x$ is the real axis, and $\mathcal{C} \longrightarrow S^1_\tau(\beta) \times \mathbb{R}_x$ is a cylinder.
- The operators $\{\hat{\phi}_x(\tau)\} \longrightarrow \hat{\phi}(x,\tau)$ are now a field.
- The Euclidean action is the integral of a Lagrange density:

$$S_{\mathrm{E}}[\{\phi_x\}] \longrightarrow S_{\mathrm{E}}[\phi] = \int_{\mathcal{C}} \mathrm{d}\tau\, \mathrm{d}x\, \mathcal{L}_{\mathrm{E}}[\phi(x,\tau)]. \tag{2.6}$$

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$$\rho_A(\phi_x, \phi'_{x'}) = \frac{1}{Z} \int_{\|,\, x \in A}^{(),\, x \notin A} \mathcal{D}\phi(y, \tau) \, e^{-S_{\mathrm{E}}}. \tag{2.7}$$

The Replica Trick

Idea: Form $\mathrm{Tr}_A(\rho_A^n)$ by gluing together n slit cylinders to create an n-sheeted *Riemann surface* $\mathcal{R}_n$. Then $\mathrm{Tr}_A(\rho_A^n)$ is the partition function of a theory identical to $\mathrm{CFT}_{\mathbb{C}}$, but defined on $\mathcal{R}_n$.

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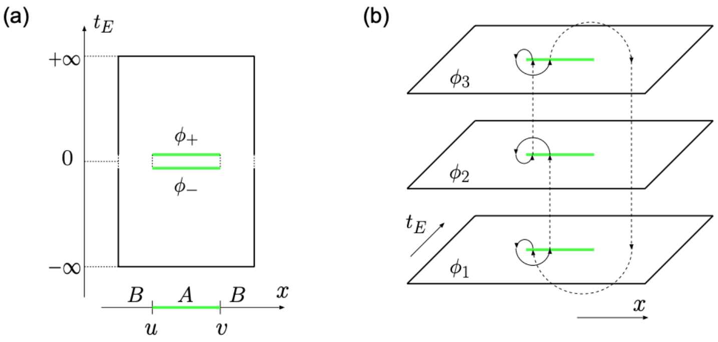

The Path Integral on $\mathcal{R}_n$

We can write this powerful result down in symbols:

Step 1: Write $\mathrm{Tr}_A(\rho_A^n)$ as a path integral on $\mathcal{R}_n$.

$$\mathrm{Tr}_A(\rho_A^n) = \frac{Z(\mathcal{R}_n)}{Z^n} = \frac{1}{Z^n} \oint_{\mathcal{R}_n} \mathcal{D}\phi \, e^{-S_{\mathrm{E}}},$$

$$S_{\mathrm{E}}[\phi] = \int_{\mathcal{R}_n} \mathrm{d}\tau \, \mathrm{d}x \, \mathcal{L}_{\mathrm{E}}[\phi(x,\tau)]. \tag{2.8}$$

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- $\mathcal{L}_{\mathrm{E}}^{\mathcal{R}_n}$ is *identical* to the original $\mathcal{L}_{\mathrm{E}}^{\mathbb{C}}$, but ϕ now lives on $\mathcal{R}_n$.
- The nontrivial topology has migrated from the BC that define $\mathrm{Tr}_A(\rho_A^n)$ to the "spacetime" domain $\mathcal{R}_n$.
- We are about to compute a partition function on a Riemann surface: this is technically applied string theory!

The Path Integral on $\mathbb{C}$: Setup

To understand the construction of $\mathrm{Tr}_A(\rho_A^n)$ in detail, it is helpful to see its index contractions written out in Einstein notation:

$$\mathrm{Tr}_A(\rho_A^n) = \rho^{\phi_x}_{\phi'_{x'}} \nearrow \rho^{\phi'_{x'}}_{\phi''_{x''}} \nearrow \cdots \nearrow \rho^{\phi^{(n)}_{x^{(n)}}}_{\phi_x} . \tag{2.9}$$

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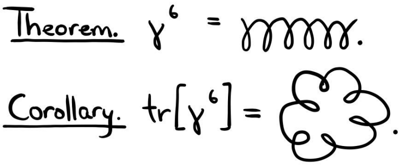
The cyclicity of the trace creates the BC, as shown in this joke:



This notation helps us write down the corresponding path integral.

The Path Integral on $\mathbb{C}$: Formulation

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$$= \frac{1}{Z^n}\int_{\mathrm{BC}} \mathcal{D}\phi_1 \cdots \mathcal{D}\phi_n \exp\left(-\int_{\mathbb{C}} \mathrm{d}\tau\, \mathrm{d}x \sum_{i=1}^{n} \mathcal{L}_{\mathrm{E}}[\phi_i(x,\tau)]\right),$$

$$\mathrm{BC}: \begin{cases} \phi_i(x,0) = \phi_i(x,\beta), & x \notin A = [u,v], \\ \phi_i(x,0) = \phi_{i+1}(x,\beta), & x \in A = [u,v]. \end{cases} \tag{2.10}$$

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But the BC prevent this from being a partition function!

Twist Fields: "Definition"

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$$\equiv \frac{1}{Z^n} \oint_{\mathbb{C}} \mathcal{D}\phi_1 \cdots \mathcal{D}\phi_n \, \Phi_n(u)\overline{\Phi}_n(v) \, e^{-S_\mathrm{E}^{(n)}[\phi_i]} =$$

$$= \left\langle \Phi_n(u)\overline{\Phi}_n(v) \right\rangle_{\mathbb{C}} = \frac{Z(\mathcal{R}_n)}{Z^n},$$

$$S_\mathrm{E}^{(n)}[\phi_1, \ldots, \phi_n] = \sum_{i=1}^{n} \int_{\mathbb{C}} \mathrm{d}\tau \, \mathrm{d}x \, \mathcal{L}_\mathrm{E}[\phi_i(x,\tau)]. \qquad (2.11)$$

Here $\langle \cdots \rangle_{\mathbb{C}}$ denotes a expectation value computed in the replica CFT, defined on $\mathbb{C}$, which has n fields and Euclidean action $S_\mathrm{E}^{(n)}$.

Twist Fields: Discussion

Ask not what twist fields *are*—ask rather what they *do*:

- Twist fields act like *defect* operators inserted at u and v; they take the place of the branch-point singularities on $\mathcal{R}_n$.

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- Twist fields implement the *monodromy* of $\mathcal{R}_n$ by permuting the ϕ_i in different directions as they pass through u or v:

$$\begin{aligned} \Phi_n(u)&: \phi_i(x,\tau) \mapsto \phi_{i+1}(x,\tau); \\ \overline{\Phi}_n(v)&: \phi_{i+1}(x,\tau) \mapsto \phi_i(x,\tau). \end{aligned} \tag{2.12}$$

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- Twist fields arise from a global internal $\mathbb{Z}_n$ symmetry: for $\sigma \in \mathbb{Z}_n$, we have $S_{\mathrm{E}}^{(n)}[\phi_1, \ldots, \phi_n] = S_{\mathrm{E}}^{(n)}[\sigma\phi_1, \ldots, \sigma\phi_n]$.
- Twist fields create conical defects, and may be used to reframe the present analysis in terms of an orbifold CFT.

A Dictionary Between Theories

From $\mathrm{Tr}_A(\rho_A^n) = \frac{Z(\mathcal{R}_n)}{Z^n} = \left\langle \Phi_n(u)\overline{\Phi}_n(v) \right\rangle_{\mathbb{C}}$, we obtain a dictionary between the CFTs on $\mathcal{R}_n$ and $\mathbb{C}$, valid for *all* correlation functions.

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If $\mathcal{O}_i(w)$ is an operator in the i^{th} copy of the replica CFT, with counterpart $\mathcal{O}(w;i)$ at $w \in \mathbb{C}_i \subset \mathcal{R}_n$ (i.e. on the i^{th} sheet), then:

$$\frac{1}{Z^n}\int_{\mathcal{R}_n} \mathcal{D}\phi\, \mathcal{O}(w;i)\, e^{-S_{\mathrm{E}}} = \left\langle \Phi_n(u)\overline{\Phi}_n(v)\, \mathcal{O}_i(w)\right\rangle_{\mathbb{C}} \implies$$

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Step 3: give a general formulation of the replica trick.

$$\langle \mathcal{O}(w;i)\rangle_{\mathcal{R}_n} = \frac{\langle \Phi_n(u)\overline{\Phi}_n(v)\, \mathcal{O}_i(w)\rangle_{\mathbb{C}}}{\langle \Phi_n(u)\overline{\Phi}_n(v)\rangle_{\mathbb{C}}}. \qquad (2.14)$$

Outline

Our Plan of Attack

Recall that correlators on $\mathcal{R}_n$ relate to those on $\mathbb{C}$ by

$$\left\langle \mathcal{O}(w;i) \right\rangle_{\mathcal{R}_n} = \frac{\left\langle \Phi_n(u)\overline{\Phi}_n(v)\, \mathcal{O}_i(w) \right\rangle_{\mathbb{C}}}{\left\langle \Phi_n(u)\overline{\Phi}_n(v) \right\rangle_{\mathbb{C}}}. \tag{3.1}$$

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We seek an operator $\mathcal{O}$ for which both the LHS and RHS can be computed. One such operator is the *stress tensor* $T^{\mu\nu}(x,\tau)$.

In "light cone" coordinates $z = x + i\tau$ and $\bar{z} = x - i\tau$, $T^{\mu\nu}(z,\bar{z})$ has 2 nonzero (anti-)holomorphic components: $T(z)$ and $\overline{T}(\bar{z})$.

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Strategy: Compute $\langle T(w) \rangle_{\mathcal{R}_n}$ by *uniformizing* $\mathcal{R}_n$ to $\mathbb{C}$. Then use the *conformal Ward identity* to understand the RHS above.

We will not need $\langle \Phi_n(u)\overline{\Phi}_n(v) \rangle_{\mathbb{C}}$, only its *scaling dimension*.

The Riemann Uniformization Theorem

A function f, analytic on $\Omega \subset \mathbb{C}$, may not be analytic on all of $\mathbb{C}$. Instead, it extends to an analytic function $\hat{f}\colon M \longrightarrow \widetilde{M}$ that "uniformizes" the Riemann surface M. If M is compact, then:

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- The image $\widetilde{M}$ of $\hat{f}$ is biholomorphic to either D^2, $\mathbb{C}$, or $\mathbb{CP}^1$. Further, $\widetilde{M}$ is the universal cover of M, and $\hat{f}$ is a lifting map.
  - If $\widetilde{M} = D^2$, then M has constant curvature -1, genus $g > 1$, and nonabelian fundamental group.
  - **If $\widetilde{M} = \mathbb{C}$, then M has constant curvature 0,** genus 1, and fundamental group $\mathbb{Z} \oplus \mathbb{Z}$.
  - If $\widetilde{M} = \mathbb{CP}^1$, then M has constant curvature $+1$, genus 0, and trivial fundamental group.
- Since $\hat{f}$ is holomorphic, it is also a conformal mapping.

Uniformizing to the Complex Plane

So f determines M, and $\hat{f}$ "uniformizes" M by realizing its image within its universal cover. We will uniformize $M = \mathcal{R}_n$ to $\widetilde{M} = \mathbb{C}$.

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$\mathcal{R}_n$ has **branch points** at $w = u$ and $w = v$. We begin by sending $(u, v) \mapsto (0, \infty)$ via the Möbius transformation $w \mapsto \zeta(w) = \frac{w-u}{w-v}$.

Next, we uniformize $\mathcal{R}_n$ to $\mathbb{C}$ using the n^{th} root, $\zeta \mapsto z(\zeta) = \zeta^{1/n}$. The uniformizing map is given by the conformal transformation

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Key idea: $\langle T \rangle$ is much easier to compute on $\mathbb{C}$ than on $\mathcal{R}_n$! All we need to know is how it transforms under conformal mappings.

Conformal Properties of $T(w)$

How does $T(w)$ transform under conformal mappings?

Theorem (BPZ, 1984)

If $w \mapsto z = f(w)$ is conformal, then $T(w)$ transforms as

$$T(w) = \left(z'(w)\right)^2 T(z) + \frac{c}{12}\{z, w\},$$
$$\{z, w\} = \frac{1}{(z')^2}\left(z'''z' - \frac{3}{2}(z')^2\right). \quad (3.3)$$

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The object $\{z, w\}$ is called the **Schwarzian derivative** of z.
It measures the failure of z to be a Möbius transformation.

The c-number c is called the **central charge** of the 2D CFT.
It measures the number of degrees of freedom in the theory. It also measures a *conformal anomaly* that prevents T from being *primary*.

The Main Calculation

Under the map $w \mapsto z(w) = \left(\frac{w-u}{w-v}\right)^{1/n}$, with $\ell = |u-v|$, we find

$$T(w;i) = \left(z'(w)\right)^2 T_i(z) + \frac{c}{12}\{z,w\} = \\ = \left[\frac{\ell}{n(w-u)(w-v)}\right]^2 \left(z^2\, T_i(z) + \frac{c}{24}(n^2-1)\right). \quad (3.4)$$

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The symmetries of $\mathbb{C}$ make $\langle T \rangle_{\mathbb{C}}$ translationally and rotationally invariant, hence constant. By finiteness of energy, it vanishes:

$$\langle T(w;i) \rangle_{\mathcal{R}_n} = \frac{c}{24}\left[\frac{\ell}{(w-u)(w-v)}\right]^2 \left(1 - \frac{1}{n^2}\right). \quad (3.5)$$

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Our dictionary relates this to correlators involving individual $T_i(z)$ in the replica theory. But we care about the *total* stress tensor of $\mathrm{CFT}^n_{\mathbb{C}}$, which is $T(z) = nT_i(z)$. Thus $\langle T(w)\rangle_{\mathcal{R}_n} = n\langle T(w;i)\rangle_{\mathcal{R}_n}$.

The Conformal Ward Identity

Step 4: Compute $\langle T \rangle_{\mathcal{R}_n}$ explicitly.

$$\frac{\left\langle \Phi_n(u)\overline{\Phi}_n(v)\, T(w)\right\rangle_{\mathbb{C}}}{\left\langle \Phi_n(u)\overline{\Phi}_n(v)\right\rangle_{\mathbb{C}}} = \frac{c}{24}\left[\frac{\ell}{(w-u)(w-v)}\right]^2\left(n-\frac{1}{n}\right). \qquad (3.6)$$

The twist fields are scalars, which fixes their 2-point functions:

$$\mathrm{Tr}_A(\rho_A^n) = \left\langle \Phi_n(u)\overline{\Phi}_n(v)\right\rangle_{\mathbb{C}} \propto |u-v|^{-2\Delta_n} = \ell^{-2\Delta_n}. \qquad (3.7)$$

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Theorem (Conformal Ward identity)

In a 2D CFT on $\mathbb{C}$ with holomorphic stress tensor $T(w)$ and a primary scalar field ϕ of scaling dimension Δ_ϕ, we have

$$\frac{\left\langle \phi(u)\phi(v)\, T(w)\right\rangle_{\mathbb{C}}}{\left\langle \phi(u)\phi(v)\right\rangle_{\mathbb{C}}} = \frac{\Delta_\phi}{2}\left[\frac{(u-v)}{(w-u)(w-v)}\right]^2. \qquad (3.8)$$

Hold On To Your Chair

Fact: Φ_n and $\overline{\Phi}_n$ are primary fields with dimensions $\Delta_n = \overline{\Delta}_n$.

We compare notes with the conformal Ward identity to obtain Δ_n:

$$\frac{\Delta_n}{2}\left[\frac{\ell}{(w-u)(w-v)}\right]^2 = \frac{c}{24}\left[\frac{\ell}{(w-u)(w-v)}\right]^2\left(n-\frac{1}{n}\right). \quad (3.9)$$

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Our computation of $\mathrm{Tr}_A(\rho_A^n)$ meets with resounding success:

Step 5: Find the scaling dimension Δ_n of the twist fields.

The Φ_n have scaling dimension $\Delta_n = \frac{c}{12}\left(n-\frac{1}{n}\right)$ and satisfy

$$\mathrm{Tr}_A(\rho_A^n) = \left\langle \Phi_n(u)\overline{\Phi}_n(v)\right\rangle_{\mathbb{C}} = c_n \ell^{-2\Delta_n} = c_n \ell^{-\frac{c}{6}\left(n-\frac{1}{n}\right)}, \quad (3.10)$$

where the c_n are undetermined constants of proportionality.

Putting the Pieces Together

We find the Rényi entropies, changing $\ell \longrightarrow \frac{\ell}{a}$ to keep S_A unitless:

$$S_A^{(n)}[\rho] = \frac{1}{1-n}\log\left[\operatorname{Tr}_A(\rho_A^n)\right] = \frac{1}{1-n}\log\left[c_n\left(\frac{\ell}{a}\right)^{-\frac{c}{6}\left(n-\frac{1}{n}\right)}\right] = \\ = \frac{c}{6}\left(1+\frac{1}{n}\right)\log\left(\frac{\ell}{a}\right) + c_n', \qquad c_n' = \frac{\log(c_n)}{1-n}. \tag{3.11}$$

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And finally, we dutifully take the limit $n \longrightarrow 1$:

$$S_A[\rho] = \lim_{n\to 1}\left[\frac{c}{6}\left(1+\frac{1}{n}\right)\log\left(\frac{\ell}{a}\right)\right] = \frac{c}{3}\log\left(\frac{\ell}{a}\right) + c_1'. \qquad (3.12)$$

Here $c_1' = \left.\frac{\partial c_n}{\partial n}\right|_{n=1}$ is an overall additive constant.

In Celebration of the Result

As we had threatened, we have derived a formula for the entropy of an interval in the ground state of a Euclidean 2D CFT on $\mathbb{C}$:

Step 6: Obtain a formula for S_A.

$$S_A[\rho] = \frac{c}{3} \log\left(\frac{\ell}{a}\right)$$

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Condensed-matter theorists hold that S_A violates the area law; high-energy theorists maintain that it doesn't. It is what it is.

Many generalizations of this result are available!

Outline

Where to Go from Here?

The state of the art allows us to say much more:

- Exact results exist at nonzero temperature or finite size.
- Systems at nonzero temperature *and* finite size are intractible in general, but some results are available in specific models.
- No exact formula exists when A consists of disjoint intervals.
- Systems with open boundary conditions, semi-infinite systems, defects, and interfaces have all been studied to some extent.
- A heuristic argument indicates thay $S_A[\rho]$ levels off to a constant in CFTs with (small) massive deformations.
- A heuristic argument describes the general appearance of logarithmic behavior in higher-dimensional CFTs.

Nonzero Temperature

Consider a conformal map from the plane $\mathbb{C}$ to a cylinder $\mathcal{C}$:

$$w \longrightarrow z(w) = \frac{\beta}{2\pi}\log(w) \text{ or } z \longrightarrow w(z) = e^{2\pi z/\beta}. \tag{4.1}$$

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The computation may be repeated as before:

$$\mathrm{Tr}_A(\rho_A^n) = \langle \Phi_n \overline{\Phi}_n \rangle_{\mathcal{C}} = c_n \left[\frac{\beta}{\pi a}\sinh\left(\frac{2\pi\ell}{\beta}\right)\right]^{-\frac{c}{6}\left(n-\frac{1}{n}\right)} \implies$$

$$S_A = \frac{c}{3}\log\left[\frac{\beta}{\pi a}\sinh\left(\frac{\pi\ell}{\beta}\right)\right]. \tag{4.2}$$

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If $\frac{\ell}{\beta} \ll 1$ (low temperature), we get $S_A = \frac{c}{3} \log\left(\frac{\ell}{a}\right)$, as expected.

If $\frac{\ell}{\beta} \gg 1$ (high temperature), we get $S_A = \frac{c\pi\ell}{3\beta} = \frac{c\pi\ell}{3} T$. This is a volume law in ℓ, and describes 1D thermal (blackbody) radiation.

Finite System Size

For a system on an interval $[0, L]$ with periodic BC, we orient the branch cut along the τ-axis at $x = 0$ and replace $\beta \longrightarrow iL$ above:

$$\mathrm{Tr}_A(\rho_A^n) = c_n \left[\frac{L}{\pi a}\sin\left(\frac{\pi \ell}{L}\right)\right]^{-\frac{c}{6}\left(n-\frac{1}{n}\right)} \Longrightarrow$$

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$$S_A = \frac{c}{3}\log\left[\frac{L}{\pi a}\sin\left(\frac{\pi\ell}{L}\right)\right]. \qquad (4.3)$$

Observe that S_A is symmetric under $\ell \longrightarrow L - \ell$.
And for $\frac{\ell}{L} \ll 1$, S_A reduces to $\frac{c}{3}\log\left(\frac{\ell}{a}\right)$, as expected.

This is practical: simulations have finite L, and can determine c for various CFTs without detailed knowledge of the spectrum.

Other Generalizations

In other directions, it is not so easy to obtain analytic results:

- **Nonzero temperature β and finite size L:** Periodic BC in both x and τ turn spacetime into a torus. Uniformization gets harder, and many CFT tools no longer work the same way.
- **N Disjoint intervals:** Making N branch cuts causes the surface $\mathcal{R}_{n,N}$ to become more complicated. It has genus $(n-1)(N-1)$, so uniformization is harder. Even for $N=2$, S_A has only been obtained in specific theories.
- **Massive deformations:** For massive QFTs with small gaps $\Delta^{-1} = \xi \gg \ell \gg a$, S_A saturates to $\frac{c}{6}\log\left(\frac{\xi}{a}\right)$. The proof parallels Zamolodchikov's proof of the c-theorem in 2D CFTs.
- **Higher dimensions:** Examples violating $S_A \sim \partial A$ are known, but an area law is expected in higher-dimensional CFTs.

A Shower Thought

The holographic interpretation S_A—in both its RT and HRT incarnations, and beyond—is a beautiful story for another time.

Recall that $\mathrm{Tr}_A(\rho_A^n) = \frac{Z(\mathcal{R}_n)}{Z^n}$ is a partition function expressible as a path integral, and the Renyi entropy $S_A^{(n)}$ is its logarithm:

$$S_A^{(n)} = \frac{1}{1-n}\log[\mathrm{Tr}_A(\rho_A^n)] = \\ = \frac{1}{(1-n)Z^n}\log\left[\int_{\mathcal{R}_n}\mathcal{D}\phi_i\, e^{-S_{\mathrm{E}}^{(n)}[\phi_i]}\right]. \qquad (4.4)$$

This $S_A^{(n)}$ looks very similar to the formal expression for an effective action in the theory of RG flows. Is $S_A^{(n)}$ as an effective action?